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Regularization of the three-dimensional gravitational potential

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Abstract. The ideas of geometrical algebra are used to investigate the physical content of the Kustaanheimo–Stiefel regularization procedure, which transforms the r^{-1} potential into the r^2 harmonic oscillator potential.

1. Introduction

The question of regularization of the gravitational potential is important and practical, because it transform the 'singular' r^{-1} potential into the well known regular one, namely r^2. This problem was solved in one dimension by Levi-Civita [1] (in 1956) who showed that the solution must proceed in two steps. First a new parameter s replaces the time variable t, then there is a 'quadratic' change of variable $r = w^2$. In the $w - s$ space the problem becomes regular. The three-dimensional generalization was obtained (in 1965) by Kustaanheimo and Stiefel [2, 3] (KS) but using an intricate matrix manipulation to extend the 'quadratic' relation in three dimensions.

On the other hand, Hestenes [4–6] has shown the power of the 'geometrical algebra' formalism (GA) which reveals a new 'deep' physical content of the usual vectors analysis. The KS formalism was recognized to be a 'manipulation' on quaternions, which appears naturally in Hestenes geometrical algebra. The aim is to show that the GA approach throws new insight onto the KS transformation, and then also on the Kepler two-body motion.

2. Geometrical algebra background

Geometrical algebra consists of building a non-commutative algebra from scalar and vectors to bivectors, and reaching finally (in three dimensions) a unique trivector†. Let us start with the three orthonormal vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$. We can define three bivectors by:

$$\vec{\boldsymbol{e}}_{12} \stackrel{\text{def}}{=} \boldsymbol{e}_1\boldsymbol{e}_2 = -\boldsymbol{e}_2\boldsymbol{e}_1 \qquad \vec{\boldsymbol{e}}_{23} \stackrel{\text{def}}{=} \boldsymbol{e}_2\boldsymbol{e}_3 = -\boldsymbol{e}_3\boldsymbol{e}_2 \qquad \vec{\boldsymbol{e}}_{31} \stackrel{\text{def}}{=} \boldsymbol{e}_3\boldsymbol{e}_1 = -\boldsymbol{e}_1\boldsymbol{e}_3 \quad (1)$$

and the unique trivector by‡:

$$\eta \stackrel{\text{def}}{=} \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3. \quad (2)$$

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† A full description can be found in [4].

‡ Here the notation η is prefered to the Hestenes one i, to enhance the 'real' character of the algebra.

In three dimensions, we have a 'duality' between bivectors and vectors through the 'volume' η

$$\vec{e}_{12} = \eta \boldsymbol{e}_3 \qquad \vec{e}_{23} = \eta \boldsymbol{e}_1 \qquad \vec{e}_{31} = \eta \boldsymbol{e}_2 \tag{3}$$

which finally gives the algebra:

	$\boldsymbol{e}_1$	$\boldsymbol{e}_2$	$\boldsymbol{e}_3$	$\eta\boldsymbol{e}_1$	$\eta\boldsymbol{e}_2$	$\eta\boldsymbol{e}_3$	η
$\boldsymbol{e}_1$	1	$\eta\boldsymbol{e}_3$	$-\eta\boldsymbol{e}_2$	η	$-\boldsymbol{e}_3$	$\boldsymbol{e}_2$	$\eta\boldsymbol{e}_1$
$\boldsymbol{e}_2$	$-\eta\boldsymbol{e}_3$	1	$\eta\boldsymbol{e}_1$	$\boldsymbol{e}_3$	η	$-\boldsymbol{e}_1$	$\eta\boldsymbol{e}_2$
$\boldsymbol{e}_3$	$\eta\boldsymbol{e}_2$	$-\eta\boldsymbol{e}_1$	1	$-\boldsymbol{e}_2$	$\boldsymbol{e}_1$	η	$\eta\boldsymbol{e}_3$
$\eta\boldsymbol{e}_1$	η	$-\boldsymbol{e}_3$	$\boldsymbol{e}_2$	-1	$-\eta\boldsymbol{e}_3$	$\eta\boldsymbol{e}_2$	$-\boldsymbol{e}_1$
$\eta\boldsymbol{e}_2$	$\boldsymbol{e}_3$	η	$-\boldsymbol{e}_1$	$\eta\boldsymbol{e}_3$	-1	$-\eta\boldsymbol{e}_1$	$-\boldsymbol{e}_2$
$\eta\boldsymbol{e}_3$	$-\boldsymbol{e}_2$	$\boldsymbol{e}_1$	η	$-\eta\boldsymbol{e}_2$	$\eta\boldsymbol{e}_1$	-1	$-\boldsymbol{e}_3$
η	$\eta\boldsymbol{e}_1$	$\eta\boldsymbol{e}_2$	$\eta\boldsymbol{e}_3$	$-\boldsymbol{e}_1$	$-\boldsymbol{e}_2$	$-\boldsymbol{e}_3$	-1

. (4)

We see that the trivector η commutes with all elements of the algebra and that the scalar and vectorial products in usual vectorial analysis VA are rewritten in GA by:

$$\begin{array}{ll} \text{VA} & \text{GA} \\ s = \boldsymbol{a} \cdot \boldsymbol{b} & s = \frac{1}{2}\{\boldsymbol{a}, \boldsymbol{b}\} \\ \boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} & \frac{1}{2}[\boldsymbol{a}, \boldsymbol{b}] = \eta \boldsymbol{c}. \end{array} \tag{5}$$

An object in the GA is then a linear combination of fundamental elements, a scalar s, a vector $\boldsymbol{v}$, a bivector $\eta\boldsymbol{w}$, and a trivector ηt:

$$O = s + \boldsymbol{v} + \eta\boldsymbol{w} + t\eta \in \mathbb{V}_0 + \mathbb{V}_1 + \mathbb{V}_2 + \mathbb{V}_3. \tag{6}$$

It is easy to see that the algebra naturally contains two subalgebras: the 'complex' algebra $\mathbb{C} = \mathbb{V}_0 + \mathbb{V}_3$ and the 'quaternion' algebra $\mathbb{Q} = \mathbb{V}_0 + \mathbb{V}_2$.

It is convenient to define two operators on the algebra defined by:

$$O^\dagger = s + \boldsymbol{v} - \eta\boldsymbol{w} - t\eta \qquad \text{and} \qquad O^* = s - \boldsymbol{v} + \eta\boldsymbol{w} - t\eta \tag{7}$$

which corresponds to permutating the vectors and to changing the sign of each vectors respectively. This unables us to extract from O each part:

$$\begin{cases} O + O^\dagger \begin{cases} (O + O^\dagger) + (O + O^\dagger)^* = 4s \\ (O + O^\dagger) - (O + O^\dagger)^* = 4\boldsymbol{v} \end{cases} \\ O - O^\dagger \begin{cases} (O - O^\dagger) + (O - O^\dagger)^* = 4\eta\boldsymbol{w} \\ (O - O^\dagger) - (O - O^\dagger)^* = 4\eta t. \end{cases} \end{cases} \tag{8}$$

3. Levi-Civita approach

The equations of motion for the $1/r$ potential can be written as†:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\boldsymbol{x} = -\frac{K}{r^3}\boldsymbol{x} \tag{9}$$

† For two bodies of mass m_1 and m_2, we have $K = G(m_1 + m_2)$ for the gravitational force and $K = \frac{e^2}{4\pi\epsilon_0}\frac{m_1+m_2}{m_1 m_2}$ for electrostatic force. $h = (E - \frac{1}{2}(m_1 + m_2)v_G^2)\frac{m_1+m_2}{m_1 m_2}$ for both, v_G is the velocity of the centre of mass.

which leads to the conservation of energy per mass:

$$h = \frac{1}{2}v^2 - \frac{K}{r}. \tag{10}$$

Let us express the time t in terms of a new variable s, we have then:

$$\begin{cases} \dfrac{1}{t'^2}\boldsymbol{x}'' - \dfrac{t''}{t'^3}\boldsymbol{x}' = -\dfrac{K}{r^3}\boldsymbol{x} \\ \dfrac{1}{2t'^2}\boldsymbol{x}'.\boldsymbol{x}' - \dfrac{K}{r} = h \end{cases} \tag{11}$$

where $'$ is the derivative by respect to s. Taking $t' = r$ leads to:

$$\begin{cases} r\boldsymbol{x}'' - r'\boldsymbol{x}' = -K\boldsymbol{x} \\ \frac{1}{2}\boldsymbol{x}'.\boldsymbol{x}' - Kr = hr^2. \end{cases} \tag{12}$$

In one dimension these equations simplify to:

$$\begin{cases} xx'' - x'x' = -Kx \\ \frac{1}{2}x'x' - Kx = hx^2. \end{cases} \tag{13}$$

The Levi-Civita proposal consists of a change of variable $x \stackrel{\text{def}}{=} w^2$ which then gives:

$$\begin{cases} 2ww'' - 2w'^2 = -K \\ 2w'^2 - K = hw^2 \end{cases} \rightarrow w'' - \frac{h}{2}w = 0. \tag{14}$$

4. The Kustaanheimo–Stiefel transformation

Following the Levi-Civita argument, we want a new variable w such that in three dimensions $\boldsymbol{x}$ be 'quadratic' in w. Furthermore, geometrical algebra shows that a pure vector is characterized by

$$O = O^\dagger \qquad \text{and} \qquad O = -O^*. \tag{15}$$

This leads to the following KS proposal:

$$\boldsymbol{x} = w^\dagger \boldsymbol{n} w \qquad \text{with } w = \alpha + \eta\boldsymbol{\beta} \qquad \text{and} \qquad \boldsymbol{nn} = 1 \tag{16}$$

where $w(s)$ belongs to the quaternion subalgebra and $\boldsymbol{n}$ is a constant unit vector. Remembering that $w^\dagger w$ is a scalar which commutes with all elements of the algebra, we also have:

$$r^2 = \boldsymbol{xx} = w^\dagger \boldsymbol{n} w w^\dagger \boldsymbol{n} w = (w^\dagger w)^2 \rightarrow w^\dagger w = \alpha^2 + \beta^2 = r. \tag{17}$$

The 'norm' of w is just the $\sqrt{r}$ as in one dimension. This enables us to write the quaternion w in terms of a unit quaternion u:

$$\begin{aligned} &w \stackrel{\text{def}}{=} \sqrt{r}\ u = \sqrt{r}(\cos\theta + \eta\sin\theta\hat{\beta}) = \sqrt{r}\mathrm{e}^{\eta\theta\hat{\beta}} \\ &\boldsymbol{x} = ru^\dagger \boldsymbol{n} u = r\mathrm{e}^{-\eta\theta\hat{\beta}}\boldsymbol{n}\mathrm{e}^{\eta\theta\hat{\beta}} \end{aligned} \tag{18}$$

($\hat{\beta}$ is the unit vector in the $\boldsymbol{\beta}$ direction).

This shows that the KS transformation consists of expressing $\boldsymbol{x}$ as a rotation θ around $\hat{\beta}$ of the vector $r\boldsymbol{n}$, and to study the dynamics of this transformation.

Futhermore, the velocity is (using $(u^\dagger u)' = 0$):

$$\left\{\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x} &= \frac{1}{r}\boldsymbol{x}' = \frac{1}{r}(r'u^\dagger \boldsymbol{n}u + ru^{\dagger'}\boldsymbol{n}u + ru^\dagger \boldsymbol{n}u') \\ &= \frac{1}{r}(r'\hat{x} + ru^{\dagger'}uu^\dagger \boldsymbol{n}u + ru^\dagger \boldsymbol{n}uu^\dagger u') \\ &= \frac{1}{r}(r'\hat{x} - ru^\dagger u'u^\dagger \boldsymbol{n}u + ru^\dagger \boldsymbol{n}uu^\dagger u') \\ &= \frac{1}{r}(r'\hat{x} - ru^\dagger u'\hat{x} + r\hat{x}u^\dagger u') \\ &= \frac{r'}{r}\hat{x} + [\hat{x}, u^\dagger u']. \end{aligned}\right. \tag{19}$$

Now, $u^\dagger u'$ is such that $(u^\dagger u')^\dagger = u^{\dagger'}u = -(u^\dagger u')$, and $(u^\dagger u')^* = (u^\dagger u')$, which means that $(u^\dagger u')$ is a pure bivector which enables us to define:

$$(u^\dagger u') \stackrel{\mathrm{def}}{=} \tfrac{1}{2}\eta\boldsymbol{\omega} \tag{20}$$

such that equation (19) reads:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x} = \frac{r'}{r}\hat{x} + \frac{1}{2}\eta[\hat{x}, \boldsymbol{\omega}] \tag{21}$$

which obviously infers that the velocity $\mathrm{d}\boldsymbol{x}/\mathrm{d}t$ is just a stretching and rotation around $\boldsymbol{\omega}$, but also that the KS transformation consists of rewriting $\boldsymbol{\omega}$ in terms of the 'product' $(u^\dagger u')$. The angular momentum (per mass) is:

$$\begin{aligned} &\eta\boldsymbol{L} \stackrel{\mathrm{def}}{=} \frac{1}{2}\left[\boldsymbol{x}, \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}\right] = \frac{\eta}{4}r[\hat{x}, [\hat{x}, \boldsymbol{\omega}]] = \frac{\eta}{4}r(4\boldsymbol{\omega} - 2\{\hat{x}, \boldsymbol{\omega}\}\hat{x}) \\ &\boldsymbol{L} = r(\boldsymbol{\omega} - \tfrac{1}{2}\{\hat{x}, \boldsymbol{\omega}\}\hat{x}) \end{aligned} \tag{22}$$

which is the orthogonal part of $\boldsymbol{\omega}$ to $\hat{x}$, $\boldsymbol{L} = r\boldsymbol{\omega}_\perp$.

We can also see that $\boldsymbol{\omega}$ rotates:

$$\tfrac{1}{2}\eta\boldsymbol{\omega} \stackrel{\mathrm{def}}{=} (u^\dagger u') = u^\dagger(u'u^\dagger)u \stackrel{\mathrm{def}}{=} \tfrac{1}{2}\eta u^\dagger \boldsymbol{\Omega} u \tag{23}$$

such that:

$$\left\{\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x} &= \frac{r'}{r}\hat{x} + \frac{1}{2}\eta[\hat{x}, u^\dagger \boldsymbol{\Omega} u] \\ &= \frac{r'}{r}\hat{x} + \frac{1}{2}\eta u^\dagger[\boldsymbol{n}, \boldsymbol{\Omega}]u \\ &= u^\dagger\left(\frac{r'}{r}\boldsymbol{n} + \frac{1}{2}\eta[\boldsymbol{n}, \boldsymbol{\Omega}]\right)u \end{aligned}\right. \tag{24}$$

which can be read as (equation (24-2)): the velocity is a composition of a stretching $\lambda\hat{x}$ and a rotation θ around $\hat{\beta}$ of the vector $\boldsymbol{n} \times \boldsymbol{\Omega}$, or (equation (24-3)) a global rotation of a 'moving' vector $\lambda\boldsymbol{n} + \boldsymbol{n} \times \boldsymbol{\Omega}$

Reviewing the velocity, in one dimension we have trivially $w^{2'} = 2ww'$ which is needed for the algebraical simplification. In three dimensions we have

$$\boldsymbol{x}' = w^{\dagger'}\boldsymbol{n}w + w^\dagger \boldsymbol{n}w' = 2w^\dagger \boldsymbol{n}w' + C_3 \tag{25}$$

where

$$C_3 \stackrel{\mathrm{def}}{=} w^{\dagger'}\boldsymbol{n}w - w^\dagger \boldsymbol{n}w' \in \mathbb{V}_3. \tag{26}$$

This also means that

$$\begin{aligned} C_3 &= r(u^{\dagger\prime}\boldsymbol{n}u - u^{\dagger}\boldsymbol{n}u') \\ &= ru^{\dagger}(uu^{\dagger\prime}\boldsymbol{n} - \boldsymbol{n}u'u^{\dagger})u \\ &= ru^{\dagger}(-u'u^{\dagger}\boldsymbol{n} - \boldsymbol{n}u'u^{\dagger})u \\ &= 2\eta r u^{\dagger}\{\boldsymbol{n}, \boldsymbol{\Omega}\}u. \end{aligned} \tag{27}$$

Given $\boldsymbol{n}$ and w we find one $\boldsymbol{x}$, while given $\boldsymbol{x}$ we have an infinite set of possibilities to choose $\boldsymbol{n}$ and w. We will then impose on $\boldsymbol{n}$ its orthogonality with $\boldsymbol{\Omega}$ such that $C_3 = 0$ and $\boldsymbol{x}' = 2w^{\dagger}\boldsymbol{n}w'$.

Choosing $C_3 = 0\ \forall s$, i.e. $\{\boldsymbol{n}, \boldsymbol{\Omega}\} = 0$ implies also that $\{\boldsymbol{x}, \boldsymbol{\omega}\} = 0$ and (see equation (22)) $\boldsymbol{L} = r\boldsymbol{\omega}$, which obviously infers that the trajectory remains orthogonal to $\boldsymbol{L}$ and that $\boldsymbol{\omega}$ is chosen in the same direction as $\boldsymbol{L}$.

5. Derivation

Let us introduce these changes of variables in the equation of motion, and in the energy equation. This leads to:

$$\begin{cases} 2rw^{\dagger\prime}\boldsymbol{n}w' + 2rw^{\dagger}\boldsymbol{n}w'' - 2r'w^{\dagger}\boldsymbol{n}w' = -Kw^{\dagger}\boldsymbol{n}w \\ 2\ w^{\dagger}\boldsymbol{n}w'\ w^{\dagger}\boldsymbol{n}w' - Kr = hr^2. \end{cases} \tag{28}$$

By trivial manipulations equation (28-1) gives:

$$2w'w^{\dagger}w' + 2rw'' - 2r'w' = -Kw \tag{29}$$

and equation (28-2)

$$2\boldsymbol{n}w'\ w^{\dagger}\boldsymbol{n}w' - Kw = hrw. \tag{30}$$

Noting that (using $C_3 = 0$):

$$r' = w^{\dagger\prime}w + w^{\dagger}w' = \boldsymbol{n}w'w^{\dagger}\boldsymbol{n} + w'w^{\dagger} \tag{31}$$

we have successively:

$$\begin{cases} 2w'w^{\dagger}w' + 2rw'' - 2r'w' = -Kw \\ 2w'w^{\dagger}w' + 2rw'' - 2(\boldsymbol{n}w'w^{\dagger}\boldsymbol{n} + w'w^{\dagger})w' = -Kw \\ 2rw'' - 2\boldsymbol{n}w'w^{\dagger}\boldsymbol{n}w' = -Kw \end{cases} \tag{32}$$

using equation (30) this finally leads to the required equation:

$$\begin{cases} 2rw'' - 2\boldsymbol{n}w'w^{\dagger}\boldsymbol{n}w' = -Kw \\ 2rw'' - hrw - Kw = -Kw \end{cases} \tag{33}$$

$$\boxed{w'' - \frac{h}{2}w = 0} \tag{34}$$

which is the regularized equation of motion!

6. Constants of motion

The linearity of the equation of motion (34) unables us to find 'trivial' conserved objects such as the Wronskian:

$$\begin{cases} C_2 = w^\dagger w' - w^{\dagger\prime} w \\ \tilde{C}_2 = w' w^\dagger - w w^{\dagger\prime} \\ C_3 = w^{\dagger\prime} \boldsymbol{n} w - w^\dagger \boldsymbol{n} w' \\ \tilde{C}_3 = w' \boldsymbol{n} w^\dagger - w \boldsymbol{n} w^{\dagger\prime} \end{cases} \tag{35}$$

$C_2, \tilde{C}_2$ are pure bivectors and $C_3, \tilde{C}_3$ are pure trivectors, all of which are conserved by the equation of motion (34). It is easy to show that:

$$C_2 = \eta r \boldsymbol{\omega} = \eta \boldsymbol{L} \qquad \text{and} \qquad \tilde{C}_2 = \eta r \boldsymbol{\Omega} \tag{36}$$

which enables us to find explicitly u:

$$\begin{cases} u^\dagger u' = \frac{1}{2}\eta \boldsymbol{\omega} \\ \dfrac{\mathrm{d}u}{\mathrm{d}s} = \dfrac{1}{2}\eta u \boldsymbol{\omega} \\ r\dfrac{\mathrm{d}u}{\mathrm{d}s} = \dfrac{1}{2}\eta u \boldsymbol{L} \end{cases} \tag{37}$$

$$u(s) = u(0)\exp\left[\frac{1}{2}\eta \boldsymbol{L}\int_0^s \frac{\mathrm{d}s}{r(s)}\right]. \tag{38}$$

We can always choose $u(0) = 1$ such that $\boldsymbol{x}(0) = r(0)\boldsymbol{n}$, and this finally gives: $\boldsymbol{\omega} = \boldsymbol{\Omega}$, and $\hat{\beta} = \hat{\omega} = \hat{\Omega}$. C_3 is chosen to be 0 and this value will remain constant. With these choices, we also have $\tilde{C}_3 = 0$

We can also build from the linear equation of motion quadratic conserved quantities. Two of them are easily obtained, a pure scalar C_0

$$C_0 \stackrel{\text{def}}{=} w^{\dagger\prime} w' - \frac{h}{2} w^\dagger w \tag{39}$$

($C_0^\dagger = C_0$ and $C_0^* = C_0$), and a pure vector $\boldsymbol{C}_1$:

$$\boldsymbol{C}_1 \stackrel{\text{def}}{=} w^{\dagger\prime} \boldsymbol{n} w' - \frac{h}{2} w^\dagger \boldsymbol{n} w \tag{40}$$

($C_1^\dagger = C_1$ and $C_1^* = -C_1$), both are conserved and are related to:

$$\begin{aligned} C_0 &= w^{\dagger\prime} w' - \frac{h}{2} w^\dagger w \\ &= \frac{1}{r} w^\dagger \boldsymbol{n} w' w^\dagger \boldsymbol{n} w' - \frac{h}{2} w^\dagger w \\ &= \frac{r}{4}\boldsymbol{v}\boldsymbol{v} - \frac{h}{2} r \\ &= \frac{K}{2} \end{aligned} \tag{41}$$

and:

$$\boldsymbol{C}_1 = w^{\dagger\prime} \boldsymbol{n} w' - \frac{h}{2} w^\dagger \boldsymbol{n} w$$

$$= \frac{1}{2}\left(\frac{1}{4}[\boldsymbol{v}, [\boldsymbol{x}, \boldsymbol{v}]] + \frac{K}{r}\boldsymbol{x}\right)$$
$$= \frac{1}{4}\boldsymbol{v}\boldsymbol{x}\boldsymbol{v} - \frac{h}{2}\boldsymbol{x}$$
$$= -\frac{1}{2}\boldsymbol{x}\left(h + \frac{v^2}{2}\right) + \frac{1}{4}\boldsymbol{v}\{\boldsymbol{x}, \boldsymbol{v}\} \tag{42}$$

which is the famous 'Laplace–Lentz' conserved vector for the $1/r$ potential! It is also easy to see that:

$$C_0^2 - C_1^2 = \frac{h}{8}([\boldsymbol{x}, \boldsymbol{v}])^2 = -\frac{1}{2}hL^2. \tag{43}$$

7. Geometrical solution

Looking at C_0 and $\boldsymbol{C}_1$, we can define a new vector $\boldsymbol{y}$ (for $h \neq 0$) such that:

$$\frac{h}{2}\boldsymbol{y} \stackrel{\text{def}}{=} w'^{\dagger}\boldsymbol{n}w' \tag{44}$$

$\boldsymbol{y}$ is, like $\boldsymbol{x}$, a solution of the equation of motion (34), and C_0, $\boldsymbol{C}_1$ can be written as:

$$\begin{cases} \dfrac{2}{|h|}C_0 = |\boldsymbol{y}| - \dfrac{h}{|h|}|\boldsymbol{x}| \\ \dfrac{2}{h}\boldsymbol{C}_1 = \boldsymbol{y} - \boldsymbol{x} \end{cases} \tag{45}$$

which can be reread as: the trajectory is on a curve such that the sum ($h < 0$) or the difference ($h > 0$) of the distances from the two extremal points of the constant vector $2\boldsymbol{C}_1/h$ is a constant. These are the usual definitions of the ellipse and hyperbola respectively with the foci located at the extremal point of $2\boldsymbol{C}_1/h$. In classical geometry (see figure 1) this means that:

$$2a = 2\frac{C_0}{|h|} \qquad \text{and} \qquad 2c = \left|2\frac{\boldsymbol{C}_1}{h}\right| \rightarrow e \stackrel{\text{def}}{=} \frac{c}{a} = \frac{|\boldsymbol{C}_1|}{C_0} \tag{46}$$

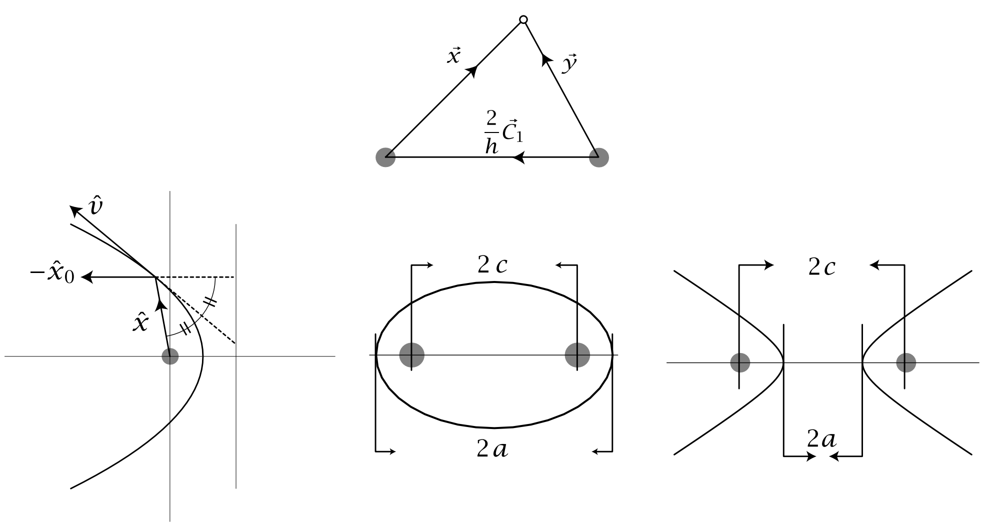

Figure 1.

which are then related directly to physical quantities by (we choose as initial condition $\{\boldsymbol{x}_0, \boldsymbol{v}_0\} = 0$):

$$a = \frac{K}{2|h|} \qquad c = r_0 \left| \frac{h + \frac{v_0^2}{2}}{2h} \right| \qquad e = \left| \frac{\frac{v_0^2}{2} + h}{\frac{v_0^2}{2} - h} \right| \tag{47}$$

and, from equation (43):

$$a^2(1-e^2) = -\frac{L^2}{2h} \to \begin{cases} h > 0 & e^2 > 1 & \text{hyperbola} \\ h < 0 & e^2 < 1 & \text{ellipse} \\ h = 0 & e^2 = 1 & \text{parabola.} \end{cases} \tag{48}$$

When $h = 0$, we have (equation (43)) $|\boldsymbol{C}_1| = C_0$, thus $e = 1$, so only the direction of $\boldsymbol{C}_1$ is a relevant quantity, we have (with $\{\boldsymbol{x}_0, \boldsymbol{v}_0\} = 0$):

$$\hat{C}_1 = \hat{v}\hat{x}\hat{v} = -\hat{x}_0 \tag{49}$$

which means that the symmetric of $\hat{x}$ with respect to $\hat{v}$ remains constant and equal to $-\hat{x}_0$. This is the 'focal' property of the parabola which states that: each vector from the focus to a point on the parabola is 'reflected' in a same direction, namely $-\hat{x}_0$, its axis.

8. Explicit solution

Let us now write the solution for $w(s)$.

- $h < 0$, $h \stackrel{\text{def}}{=} -2\gamma^2$

$$\begin{aligned} w(s) &= w_0 \cos(\gamma s) + w_0' \frac{\sin(\gamma s)}{\gamma} \\ w'(s) &= -\gamma w_0 \sin(\gamma s) + w_0' \cos(\gamma s). \end{aligned} \tag{50}$$

Let us take, as initial condition $\{\boldsymbol{x}_0, \boldsymbol{v}_0\} = 0$, such that the initial velocity is orthogonal to the position, then:

$$\boldsymbol{C}_1 = \gamma^2 \boldsymbol{x}_0 \left(1 - \frac{v_0^2}{4\gamma^2}\right) \tag{51}$$

$$\begin{cases} r(s) = w^\dagger(s) w(s) \\ \quad = \cos^2(\gamma s) w_0^\dagger w_0 + \dfrac{\cos(\gamma s)\sin(\gamma s)}{\gamma}(w_0^\dagger w_0' + w'^\dagger_0 w_0) + \dfrac{\sin^2(\gamma s)}{\gamma^2} w'^\dagger_0 w_0' \\ \quad = \cos^2(\gamma s) r_0 + \dfrac{\sin(2\gamma s)}{4\gamma}\{\boldsymbol{x}_0, \boldsymbol{v}_0\} + \dfrac{\sin^2(\gamma s)}{4\gamma^2} r_0 v_0^2 \\ \quad = \dfrac{r_0}{2}\left(1 + \dfrac{v_0^2}{4\gamma^2}\right) + \cos(2\gamma s)\dfrac{r_0}{2}\left(1 - \dfrac{v_0^2}{4\gamma^2}\right) \end{cases} \tag{52}$$

$$r(s)' = \frac{1}{2}\{\boldsymbol{x}(s), \boldsymbol{v}(s)\} = -\gamma \sin(2\gamma s) r_0 \left(1 - \frac{v_0^2}{4\gamma^2}\right) \tag{53}$$

given then the time t versus s (taking $t = 0$ when $s = 0$):

$$t(s) = \frac{r_0}{2}\left(1 + \frac{v_0^2}{4\gamma^2}\right) s + \sin(2\gamma s)\frac{r_0}{4\gamma}\left(1 - \frac{v_0^2}{4\gamma^2}\right) \tag{54}$$

finally:

$$\begin{aligned}\boldsymbol{x}(s) = w^{\dagger}(s)\boldsymbol{n}w(s) &= \frac{1}{2}\boldsymbol{x}_0\left(\left(1-\frac{v_0^2}{4\gamma^2}\right)+\cos(2\gamma s)\left(1+\frac{v_0^2}{4\gamma^2}\right)\right)\\ &\quad+\frac{1}{2\gamma}r_0\sin(2\gamma s)\boldsymbol{v}_0\end{aligned} \tag{55}$$

$$\boldsymbol{v} = \frac{1}{r(s)}\left(-\gamma\boldsymbol{x}_0\sin(2\gamma s)\left(1+\frac{v_0^2}{4\gamma^2}\right)+r_0\cos(2\gamma s)\boldsymbol{v}_0\right). \tag{56}$$

This shows an elliptic motion with principal axis length $2a$ and excentricity e given by:

$$2a = r_0\left(1+\frac{v_0^2}{4\gamma^2}\right) = \frac{K}{2\gamma^2} \qquad e = 1-\frac{4\gamma^2 r_0}{K}. \tag{57}$$

A periodic motion in s and t with periodicity Δs, Δt:

$$\Delta s = \frac{\pi}{\gamma} \rightarrow \Delta t = \frac{r_0}{2}\left(1+\frac{v_0^2}{4\gamma^2}\right)\frac{\pi}{\gamma} = \frac{K\pi}{4\gamma^3} = \frac{K\pi}{\sqrt{2|h|^3}} = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{K}} \tag{58}$$

showing the Kepler proportionality between Δt^2 and a^3

- $h > 0$, $h \stackrel{\text{def}}{=} 2\gamma^2$

$$\boldsymbol{C}_1 = -\gamma^2\boldsymbol{x}_0\left(1+\frac{v_0^2}{4\gamma^2}\right) \tag{59}$$

$$\begin{aligned}w(s) &= w_0\cosh(\gamma s)+w_0'\frac{\sinh(\gamma s)}{\gamma}\\ w'(s) &= \gamma w_0\sinh(\gamma s)+w_0'\cosh(\gamma s)\end{aligned} \tag{60}$$

which gives the hyperbolic motion (with $\{\boldsymbol{x}_0, \boldsymbol{v}_0\} = 0$):

$$\begin{aligned}r(s) &= \frac{r_0}{2}\left(1-\frac{v_0^2}{4\gamma^2}\right)+\frac{r_0}{2}\cosh(2\gamma s)\left(1+\frac{v_0^2}{4\gamma^2}\right)\\ t(s) &= \frac{r_0}{2}\left(1-\frac{v_0^2}{4\gamma^2}\right)s+\frac{r_0}{4\gamma}\sinh(2\gamma s)\left(1+\frac{v_0^2}{4\gamma^2}\right)\\ \boldsymbol{x}(s) &= \frac{1}{2}\boldsymbol{x}_0\left(\left(1+\frac{v_0^2}{4\gamma^2}\right)+\cosh(2\gamma s)\left(1-\frac{v_0^2}{4\gamma^2}\right)\right)+\frac{1}{2\gamma}r_0\sinh(2\gamma s)\boldsymbol{v}_0\\ \boldsymbol{v} &= \frac{1}{r(s)}\left(\gamma\boldsymbol{x}_0\sinh(2\gamma s)\left(1-\frac{v_0^2}{4\gamma^2}\right)+r_0\cosh(2\gamma s)\boldsymbol{v}_0\right).\end{aligned} \tag{61}$$

- $h = 0$.

Finally, the parabolic cases give respectively (again with the initial orthogonality $\{\boldsymbol{x}_0, \boldsymbol{v}_0\} = 0$):

$$w(s) = w_0 + w_0' s \qquad \boldsymbol{C}_1 = -\tfrac{1}{4}\boldsymbol{x}_0 v_0^2 \tag{62}$$

so:

$$
\begin{aligned}
r(s) &= r_0 + \frac{r_0 v_0^2}{4} s^2 \\
t(s) &= r_0 s + \frac{r_0 v_0^2}{12} s^3 \\
\boldsymbol{x}(s) &= \boldsymbol{x}_0 \left(1 - \frac{v_0^2}{4} s^2\right) + r_0 \boldsymbol{v}_0 s \\
\boldsymbol{v} &= \frac{\boldsymbol{v}_0 - \frac{\boldsymbol{x}_0}{r_0} \frac{v_0^2}{2} s}{1 + \frac{v_0^2}{4} s^2}.
\end{aligned}
\tag{63}
$$

9. Conclusions

The geometrical algebra approach to the motion of a two-body system interacting through a $1/r$ potential, gives rise to numerous and fruitful analyses and discussions. The simplification induced by the 'elementary' algebra treatment of the problem enables us to derive the geometrical properties of the curves through physical parameters, and to concentrate our attention on the physical meaning of 'objects' that we introduce. Analytical solutions are then obtained using elementary analysis.

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